

SELF-SIMILAR MINIMIZERS OF A BRANCHED TRANSPORT FUNCTIONAL

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ABSTRACT. We solve here completely an irrigation problem from a Dirac mass to the Lebesgue measure. The functional we consider is a two dimensional analog of a functional previously derived in the study of branched patterns in type-I superconductors. The minimizer we obtain is a self-similar tree.

1. INTRODUCTION

In this paper we consider the functional (see Section 2 for a more precise definition)

$$(1.1) \quad \mathcal{E}(\mu) := \int_a^b \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt,$$

where $\mu = \mu_t \otimes dt$ with $\mu_t = \sum_i \varphi_i \delta_{X_i}$ for a.e. $t \in (a, b)$ for some (pairwise distinct) $X_i \in \mathbb{R}$ and where \dot{X}_i denotes the time derivative of $X_i(t)$. For two given measures μ_{\pm} of equal mass, we study the following Dirichlet problem

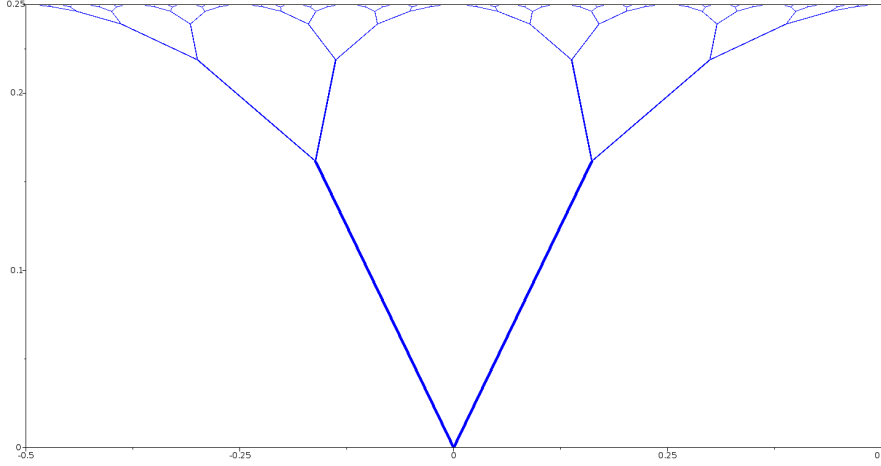
$$(1.2) \quad \min_{\mu} \{\mathcal{E}(\mu) : \mu_a = \mu_- , \mu_b = \mu_+\}.$$

Our main result is a full characterization of the minimizers of (1.2) in the case μ_- is a Dirac mass, μ_+ is the Lebesgue measure restricted to an interval of length $\mu_-(\mathbb{R})$ and $b - a$ is large enough. In order to fix notation, since the problem is invariant by translations, we may assume that $a = 0$, $b = T$, $\mu_- = \varphi \delta_X$ and $\mu_+ = dx \llcorner [-\varphi/2, \varphi/2]$ for some $T, \varphi > 0$ and $X \in \mathbb{R}$. As will be apparent below, up to rescalings and shears, we may further normalize to $X = 0$ and $\varphi = 1$, so that

$$\mu_0 = \delta_0 \quad \text{and} \quad \mu_T = dx \llcorner [-1/2, 1/2].$$

In this case, as will become clearer in the proof, the threshold value $T = 1/4$ naturally appears. In order to state our main theorem, let us define for $t \in [0, 1/4]$, the measure μ_t^* (see Figure 1). For $k \geq 0$, let $t_k := \frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^{3^{k/2}}\right)$. We define recursively μ_t^* in the intervals $[t_{k-1}, t_k]$. Let $X_1^0 = 0$ and $\mu_0^* = \delta_0$. Assume that μ_t^* is defined in $[0, t_{k-1}]$ and that $\mu_{t_{k-1}}^* = 2^{-(k-1)} \sum_{i=1}^{2^{k-1}} \delta_{X_i^{k-1}}$. For $t \in [t_{k-1}, t_k]$ and $1 \leq i \leq 2^k$, we now define $X_i^k(t)$. For this, let us divide $[-1/2, 1/2]$ in 2^k intervals of equal size and let \bar{X}_i^k be the barycenter of

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FIGURE 1. The optimal configuration μ^*

the i -th such interval i.e. $\bar{X}_i^k := \frac{-1}{2} + \frac{i-1}{2^k} + \frac{1}{2^{k+1}}$. We then let

$$X_i^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}} \left(\bar{X}_i^k - X_{[i/2]}^{k-1} \right) + X_{[i/2]}^{k-1},$$

and $\mu_t^* := 2^{-k} \sum_{i=1}^{2^k} \delta_{X_i^k(t)}$. Notice that with this definition, for every $t \in [0, 1/4)$, $k \in \mathbb{N}$ and $1 \leq i \leq 2^k$, the mass at $X_i^k(t)$ is irrigating the interval $(\bar{X}_i^k - 2^{-(k+1)}, \bar{X}_i^k + 2^{-(k+1)})$ and X_i^k is moving at constant speed towards \bar{X}_i^k (and would reach it at time $T = 1/4$ if there were no further branching points). Our main theorem is the following

Theorem 1.1. *For $T = 1/4$, $\mu_0 = \delta_0$ and $\mu_T = dx \llcorner [-1/2, 1/2]$, μ^* is the unique minimizer of (1.2). Moreover, if $T \geq 1/4$, the unique minimizer of (1.2) is given by $\mu_t = \delta_0$ for $t \in [0, T - 1/4]$ and $\mu_t = \mu_{t-(T-1/4)}^*$ for $t \in (T - 1/4, T)$, with*

$$\mathcal{E}(\mu) = \frac{1}{2 - \sqrt{2}} + T.$$

As a consequence, we obtain the following corollary (see Lemma 3.1 for the exact definitions of the rescaling and shear)

Corollary 1.2. *For $X \in \mathbb{R}$, $T, \varphi > 0$ with $T\varphi^{-3/2} \geq 1/4$, the unique minimizer of (1.2) with $\mu_0 = \varphi\delta_X$ and $\mu_T = dx \llcorner [-\varphi/2, \varphi/2]$ is given by a suitably sheared and rescaled version of the optimal measure for $X = 0$, $\varphi = 1$ and $T' = T\varphi^{-3/2}$. Moreover*

$$\mathcal{E}(\mu) = \varphi^{3/2} \frac{1}{2 - \sqrt{2}} + T + \frac{\varphi}{T} |X|^2.$$

As an application of Corollary 1.2, we will further derive a full characterization of symmetric (with respect to $t = 0$) minimizers in the case $a = -b = -T$, $\mu_{\pm} = dx \llcorner [-1/2, 1/2]$

and $T \geq 1/4$ (see Theorem 4.1).

The proof of Theorem 1.1 is based on the tree structure of the minimizers of (1.2) (see Proposition 2.7) which together with invariance by scaling and shearing (Lemma 3.1) leads to a recursive characterization of the minimizers (see (3.4)). This in turn, allows for the first branching time $T = 1/4$, to rewrite the geometrical problem (1.2) in term of a purely analytical minimization problem (see (3.12)). Using a computer assisted proof, this permits to exclude the possibility of branching of three or more branches (see Section 3.1). Once it is known that only pairs of branches appear at each branching point, it is possible to prove that mass splits always in half and to compute the distance between branching points (see Section 3.2).

The variational problem (1.2) may be seen as a two dimensional (one for time and one for space) analog of the three dimensional (one for time and two for space) problem derived in [8] as a reduced model for the description of branching in type-I superconductors in the regime of very small applied external field. We refer the reader to [8] for more precise physical motivations and references. In this regime, the natural Dirichlet conditions appearing are $\mu_{\pm} = dx \llcorner [-1/2, 1/2]$. Let us point out that in the three dimensional model, the term $\#\{\varphi_i \neq 0\}$ is replaced by $\sum_i \varphi_i^{1/2}$. This is in line with the interpretation of the first term in (1.1) as an interfacial term penalizing the creation of many flux tubes. That is, if we are in $(1+d)$ -dimensions, it is proportional to the perimeter of a union of d -dimensional balls of volume φ_i (which is $2\sqrt{\pi} \sum_i \varphi_i^{1/2}$ if $d = 2$ and $2\#\{\varphi_i \neq 0\}$ if $d = 1$). The second term in (1.1) may be interpreted as the Wasserstein transportation cost of moving such balls. In many models describing pattern formation in material sciences, branching patterns similar to the one observed here are expected. However, it is usually very hard to go beyond scaling laws [10, 16, 5, 9]. In some cases, reduced models have been derived [12, 8, 7] but so far the best results concerning the minimizers are local energy bounds leading to the proof of asymptotic self-similarity [6, 14]. Our result is thus the first complete characterization of a minimizer in this context. Of course, this was possible thanks to the simplicity of our model (one dimensional trees in a two dimensional ambient space). We should however point out that our result is not fully satisfactory since we are essentially able to study only the situation of an isolated microstructure (due to the constraint $T \geq 1/4$) whereas one is typically interested in the case $T \ll 1$ where many microstructures are present and where the lateral boundary conditions have limited effect i.e. one tries to capture an extensive behavior of the system. As detailed in the final section 4, we believe that even in the regime $T \ll 1$, every microstructure is of the type described in Corollary 1.2.

As pointed out in [8], the functional (1.1) bears many similarities with so-called branched transport (or irrigation) models [2]. Also in this class of problems, there has been a strong interest for the possible fractal behavior of minimizers. Besides results on scaling laws [4] and fractal regularity [3], to the best of our knowledge, the only explicit minimizers

exhibiting infinitely many branching points have been obtained in [13] for a Dirac irrigating a Cantor set and in [11] in an infinite dimensional context. In particular, the optimal irrigation pattern from a Dirac mass to the Lebesgue measure is currently not known for the classical branched transportation model. One important difference between our model and branched transportation is that in our case, minimality does not imply triple junctions nor conditions on the angles between the branches.

The organization of the paper is the following. In Section 2, we recall the definition and the basic properties of the functional \mathcal{E} . Then, in Section 3 we prove Theorem 1.1. In the final Section 4, we give an application of Theorem 1.1 to the irrigation of the Lebesgue measure by itself and state an open problem.

Notation In the paper we will use the following notation. The symbols $\simeq, \gtrsim, \lesssim, \ll$ indicate estimates that hold up to a global constant. For instance, $f \lesssim g$ denotes the existence of a constant $C > 0$ such that $f \leq Cg$. We denote by \mathcal{H}^1 the 1-dimensional Hausdorff measure. For a Borel measure μ , we will denote by $\text{supp } \mu$ its support.

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2. THE VARIATIONAL PROBLEM AND MAIN PROPERTIES OF THE FUNCTIONAL

Definition 2.1. For $a < b$ we denote by $\mathcal{A}_{a,b}$ the set of pairs of measures $\mu \geq 0$, m with $m \ll \mu$, satisfying the continuity equation

$$(2.1) \quad \partial_t \mu + \partial_x m = 0 \quad \text{in } \mathbb{R} \times (a, b)$$

and such that $\mu = \mu_t \otimes dt$ where, for a.e. $t \in (a, b)$, $\mu_t = \sum_i \varphi_i \delta_{X_i}$ for some $\varphi_i \geq 0$ and $X_i \in \mathbb{R}$. We denote by $\mathcal{A}_{a,b}^* := \{\mu : \exists m, (\mu, m) \in \mathcal{A}_{a,b}\}$ the set of admissible μ .

Further, we define $\mathcal{E} : \mathcal{A}_{a,b} \rightarrow [0, \infty]$ by

$$(2.2) \quad \mathcal{E}(\mu, m) := \int_a^b \# \{\text{supp } \mu_t\} dt + \int_{\mathbb{R} \times (a,b)} \left(\frac{dm}{d\mu} \right)^2 d\mu$$

and (with abuse of notation) $\mathcal{E} : \mathcal{A}_{a,b}^* \rightarrow [0, \infty]$ by

$$(2.3) \quad \mathcal{E}(\mu) := \min \{ \mathcal{E}(\mu, m) : m \ll \mu, \partial_t \mu + \partial_x m = 0 \}.$$

Equation (2.1) is understood in the sense of distributions (testing with test functions in $C_c^\infty(\mathbb{R} \times (0, T))$). Contrary to [8], we use free boundary conditions instead of periodic ones but this makes only minor differences. In the sequel we will only deal with measures μ of bounded support. In this case, because of (2.1), $\mu_t(\mathbb{R})$ does not depend on t . Let

us point out that for such measures, the minimum in (2.3) is attained thanks to [1, Th. 8.3.1]. Moreover, the minimizer is unique by strict convexity of $m \rightarrow \int_{\mathbb{R} \times (a,b)} \left(\frac{dm}{d\mu}\right)^2 d\mu$. Let us also notice that by the Benamou-Brenier formula [1], we have for every measure μ , and every $t, t' \in (a, b)$,

$$(2.4) \quad W_2^2(\mu_t, \mu_{t'}) \leq \mathcal{E}(\mu)|t - t'|,$$

where the 2-Wasserstein distance between two measures μ and ν of bounded second moment with $\mu(\mathbb{R}) = \nu(\mathbb{R})$ is defined by

$$W_2^2(\mu, \nu) := \min \left\{ \int_{\mathbb{R} \times \mathbb{R}} |x - y|^2 d\Pi(x, y) : \Pi_1 = \mu, \Pi_2 = \nu \right\},$$

where the minimum is taken over measures on $\mathbb{R} \times \mathbb{R}$ and Π_1 and Π_2 are respectively the first and second marginal of Π . In particular for every measure μ with $\mathcal{E}(\mu) < \infty$, the curve $t \mapsto \mu_t$ is Hölder continuous with exponent one half in the space of measures (endowed with the metric W_2) and the traces μ_a and μ_b are well defined.

Given two measures μ_{\pm} on \mathbb{R} with $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R})$ and bounded support, we are interested in the variational problem

$$(2.5) \quad \inf \{ \mathcal{E}(\mu) : \mu_a = \mu_-, \mu_b = \mu_+ \}.$$

Let us first notice that if $L > 0$ is such that $\text{supp } \mu_- \cup \text{supp } \mu_+ \subseteq [-L/2, L/2]$, then we may restrict the infimum in (2.5) to measures satisfying $\text{supp } \mu_t \subseteq [-L/2, L/2]$ for a.e. $t \in (a, b)$. Indeed, if μ is admissible with $\mu_t = \sum_i \varphi_i \delta_{X_i}$ then letting $\tilde{X}_i := \min(L/2, |X_i|) \text{sign } X_i$ and then $\tilde{\mu}_t := \sum_i \varphi_i \delta_{\tilde{X}_i}$, we get that $\tilde{\mu}$ is admissible and has lower energy than μ (i.e. the energy decreases by projection on $[-L/2, L/2]$). From now on we will only consider such measures.

As in [8, Prop 5.2] (to which we refer for the proof), a simple branching construction shows that any pair of measures with equal flux may be connected with finite cost.

Proposition 2.2. *For every pair of measures μ_{\pm} with $\text{supp } \mu_{\pm} \subseteq [-L/2, L/2]$ and $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R}) = \varphi$, there is $\mu \in \mathcal{A}_{a,b}^*$ such that letting $b - a = 2T$, $\mu_a = \mu_-$, $\mu_b = \mu_+$ and*

$$\mathcal{E}(\mu) \lesssim T + \frac{\varphi L^2}{T}.$$

If $\mu_+ = \mu_-$, then there is a construction with

$$\mathcal{E}(\mu) \lesssim T + T^{1/3} \varphi^{1/3} L^{2/3}.$$

From this, arguing as in [8, Prop. 5.5], we obtain that

Proposition 2.3. *For every pair of measures μ_{\pm} with bounded support and $\mu_+(\mathbb{R}) = \mu_-(\mathbb{R})$, the infimum in (2.5) is finite and attained.*

We now give some regularity results for minimizers of (2.5). These can be mostly proven as in [8] so we state them without proof. Let us first recall the notion of subsystem.

Proposition 2.4 (Definition of a subsystem). *Given a point $(X, t) \in [-L/2, L/2] \times (a, b)$ and $\mu \in \mathcal{A}_{a,b}^*$ with $\mathcal{E}(\mu) < \infty$, there exists a subsystem μ' of μ emanating from (X, t) in the sense that there exists μ' such that*

- (i) $\mu' \leq \mu$ in the sense that $\mu - \mu'$ is a positive measure,
- (ii) $\mu'_t = a\delta_X$, where $a = \mu_t(X)$,
- (iii) if m is such that $\mathcal{E}(\mu) = \mathcal{E}(\mu, m)$, then

$$\partial_t \mu' + \partial_x \left(\frac{dm}{d\mu} \mu' \right) = 0.$$

In particular, (ii) implies that $(\mu_t - \mu'_t) \perp \delta_X$ in the sense of the Radon-Nikodym decomposition. We call $\mu^+ := \mu' \llcorner \mathbb{R} \times (t, b)$ the forward subsystem emanating from (X, t) and $\mu^- := \mu' \llcorner \mathbb{R} \times (a, t)$ the backward subsystem emanating from X .

Lemma 2.5 (No loops). *Let μ be a minimizer for the Dirichlet problem (2.5), $\bar{t} \in (a, b)$. Let X_1, X_2 be two points in the line $\{(x, t) : t = \bar{t}\}$. Let μ_1 and μ_2 be subsystems of μ emanating from (X_1, \bar{t}) , resp. (X_2, \bar{t}) . Let (X_+, t_+) be a point with $t_+ > \bar{t}$ and (X_-, t_-) a point with $t_- < \bar{t}$, and such that μ_1 and μ_2 both have Diracs at both X_+ and X_- with nonzero mass. Then $X_1 = X_2$.*

As in [8], a consequence of this lemma is that we have a representation of the form

$$(2.6) \quad \mu = \sum_i \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i$$

where the sum is countable and $\Gamma_i = \{(X_i(t), t) : t \in [a_i, b_i]\}$ where X_i are absolutely continuous and almost everywhere non overlapping.

Another consequence is that if there are two levels at which μ is a finite sum of Diracs, then it is the case for all the levels in between. Since $\mathcal{E}(\mu) < \infty$ implies in particular that $\#\{\varphi_i \neq 0\} < \infty$ for a.e. $t \in (a, b)$, this means that μ_t is in fact a locally finite (in time) sum of Dirac masses and thus, the sum in (2.6) is finite away from the initial and finite time.

For measures which are concentrated on finitely many curves, we have as in [8, Lem. 5.9], a representation formula for $\mathcal{E}(\mu)$.

Lemma 2.6. *Let $\mu = \sum_{i=1}^N \frac{\varphi_i}{\sqrt{1 + |\dot{X}_i|^2}} \mathcal{H}^1 \llcorner \Gamma_i \in \mathcal{A}_{a,b}^*$ with $\Gamma_i = \{(X_i(t), t) : t \in [a_i, b_i]\}$ for some absolutely continuous curves X_i , disjoint up to the endpoints. Every φ_i is then constant on $[a_i, b_i]$ and we have conservation of mass. That is, for $z := (x, t)$, letting*

$$\begin{aligned} \mathcal{I}^-(z) &:= \{i \in [1, N] : t = b_i, X_i(b_i) = x\} \\ \mathcal{I}^+(z) &:= \{i \in [1, N] : t = a_i, X_i(a_i) = x\}, \end{aligned}$$

it holds

$$\sum_{i \in \mathcal{I}^-(z)} \varphi_i = \sum_{i \in \mathcal{I}^+(z)} \varphi_i.$$

Moreover, $m = \sum_i \frac{\varphi_i}{\sqrt{1+|\dot{X}_i|^2}} \dot{X}_i \mathcal{H}^1 \llcorner \Gamma_i$ and

$$(2.7) \quad \mathcal{E}(\mu) = \sum_i \int_{a_i}^{b_i} 1 + \varphi_i |\dot{X}_i|^2 dt.$$

In particular, this proves that for minimizers, formula (2.7) holds (where the sum is at most countable). By a slight abuse of notation, for such measures we will denote

$$\mathcal{E}(\mu) = \int_a^b \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt.$$

We gather below some properties of the minimizers

Proposition 2.7. *A minimizer of the Dirichlet problem (2.5) with boundary conditions μ_{\pm} satisfies*

- ((i)) *Each X_i is affine.*
- ((ii)) *There is monotonicity of the traces in the sense that for every $t \in (a, b)$, if $\mu_t = \sum_i \varphi_i \delta_{X_i}$ with X_i ordered (i.e. $X_i \leq X_{i+1}$) and if $\mu^{i,+}$ is the forward subsystem emanating from X_i , then the traces $\mu_b^{i,+}$ satisfy $\text{supp } \mu_b^{i,+} = [x_i^+, y_i^+]$ with $y_i^+ \leq x_{i+1}^+$. The analogous statement holds for the backward subsystems.*
- ((iii)) *If $\mu_- = \varphi \delta_X$ then μ has a tree structure.*
- ((iv)) *If $\mu_- = \mu_+$, then letting $a = -T$ and $b = T$, there exists a minimizer which is symmetric with respect to the $t = 0$ plane. For every such minimizer, the number of Dirac masses at time t is minimal for $t = 0$.*

Proof. Item (i) follows from fixing the branching points and minimizing in X_i . The other points are simple consequences of Lemma 2.5. \square

The monotonicity property (ii), is analogous to the monotonicity of optimal transport maps in one space dimension [15]. As a consequence of (iv), in the case $\mu_{\pm T} = \varphi/L dx \llcorner [-L/2, L/2]$, we may get an estimate on the number of Dirac masses on the mid-plane $t = 0$.

Lemma 2.8. *For $T \ll \varphi^{1/2}L$, if μ is a symmetric minimizer of*

$$\min\{\mathcal{E}(\mu) : \mu_{\pm T} = \varphi/L dx \llcorner [-L/2, L/2]\}$$

and if $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$, then $N \simeq \varphi^{1/3} L^{2/3} T^{-2/3}$ and the number N_{good} of φ_i satisfying $\varphi_i \simeq (T\varphi L^{-1})^{2/3}$ is of order N .

Proof. Since μ is symmetric, by Lemma 2.7, letting for $i = 1, \dots, N$, $\mu^{i,+}$ be the forward subsystem emanating from X_i , we have by (2.4),

$$\mathcal{E}(\mu) \gtrsim NT + \frac{1}{T} \sum_{i=1}^N W_2^2(\varphi_i \delta_{X_i}, \mu_T^{i,+}) \gtrsim NT + \frac{1}{T} \sum_{i=1}^N \varphi_i^3.$$

By Proposition 2.2, we have $\mathcal{E}(\mu) \lesssim T^{1/3} \varphi^{1/3} L^{2/3}$, so that

$$T^{1/3} \varphi^{1/3} L^{2/3} \gtrsim NT + \frac{1}{T} \sum_{i=1}^N \varphi_i^3.$$

Since $\sum_{i=1}^N \varphi_i^3 \geq \frac{\varphi L^2}{N^2}$, we readily conclude the proof. \square

3. IRRIGATION OF THE LEBESGUE MEASURE BY A DIRAC MASS

In this section we consider (1.2) with $a = 0$ and $b = T$, $\mu_- = \varphi \delta_X$ and $\mu_+ = dx \llcorner [-\varphi/2, \varphi/2]$. We will denote

$$E(T, \varphi, X) := \min\{\mathcal{E}(\mu) : \mu_0 = \varphi \delta_X, \mu_T = dx \llcorner [-\varphi/2, \varphi/2]\}.$$

For simplicity we let $E(T, \varphi) := E(T, \varphi, 0)$ be the energy required to connect the Lebesgue measure to the centered Dirac mass and $E(T) := E(T, 1)$. The following lemma shows that understanding $E(T)$ is enough for understanding $E(T, \varphi, X)$.

Lemma 3.1. *For every T, φ, X , it holds,*

$$(3.1) \quad E(T, \varphi, X) = E(T, \varphi) + \frac{1}{T} \varphi |X|^2.$$

Moreover, if $\mu_t = \sum_i \varphi_i X_i(t)$ is optimal for $E(T, \varphi)$, then letting $\hat{X}_i(t) := (1 - \frac{t}{T})X + X_i(t)$, $\hat{\mu}_t = \sum_i \varphi_i \hat{X}_i(t)$ is optimal for $E(T, \varphi, X)$.

Furthermore, we have

$$(3.2) \quad E(T, \varphi) = \varphi^{3/2} E(T \varphi^{-3/2}).$$

In addition, if $\mu_t = \sum_i \varphi_i X_i(t)$ is optimal for $E(T \varphi^{-3/2})$ then letting $\hat{t} := \varphi^{3/2} t$, $\hat{\varphi}_i := \varphi \varphi_i$, and $\hat{X}_i := \varphi X_i(\hat{t})$, then $\hat{\mu}_{\hat{t}} = \sum_i \hat{\varphi}_i \delta_{\hat{X}_i}$ is optimal for $E(T, \varphi)$.

Proof. For $\mu_t = \sum_i \varphi \delta_{X_i}$ admissible for $E(T, X)$, we define $\hat{\mu}_t := \sum_i \varphi_i \delta_{\hat{X}_i}$, where $\hat{X}_i(t) := (1 - \frac{t}{T})X + X_i(t)$. Then, $\hat{\mu}_t$ is admissible for $E(T, \varphi, X)$ and

$$(3.3) \quad \begin{aligned} \mathcal{E}(\hat{\mu}) &= \int_0^T \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i - \frac{1}{T} X|^2 dt \\ &= \int_0^T \#\{\varphi_i \neq 0\} + \sum_i \varphi_i |\dot{X}_i|^2 dt + \frac{|X|^2}{T^2} \int_0^T \sum_i \varphi_i dt - 2 \frac{X}{T} \int_0^T \sum_i \varphi_i \dot{X}_i dt. \end{aligned}$$

For $\varepsilon > 0$, thanks to Lemma 2.6 and the fact that $X_i(0) = 0$, we have

$$\int_0^{T-\varepsilon} \sum_i \varphi_i \dot{X}_i dt = \sum_i \varphi_i X_i(T - \varepsilon).$$

Furthermore, testing the weak convergence of μ_t to dx as $t \rightarrow T$, with the function x , we get

$$\lim_{\varepsilon \rightarrow 0} \sum_i \varphi_i X_i(T - \varepsilon) = \int_{-\varphi/2}^{\varphi/2} x dx = 0.$$

Finally, since by Hölder's inequality applied twice and $\sum_i \varphi_i = \varphi$,

$$\int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i| \leq \varphi^{1/2} \int_{T-\varepsilon}^T (\sum_i \varphi_i |\dot{X}_i|^2)^{1/2} \leq \varphi^{1/2} \varepsilon^{1/2} \left(\int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i|^2 \right)^{1/2},$$

we get

$$\int_0^T \sum_i \varphi_i \dot{X}_i dt = \lim_{\varepsilon \rightarrow 0} \left(\int_0^{T-\varepsilon} \sum_i \varphi_i \dot{X}_i dt + \int_{T-\varepsilon}^T \sum_i \varphi_i |\dot{X}_i| dt \right) = 0.$$

Combining this with $\sum_i \varphi_i = \varphi$ and (3.3), we get

$$\mathcal{E}(\hat{\mu}) = \mathcal{E}(\mu) + \frac{1}{T} \varphi |X|^2,$$

from which the first part of the proposition follows.

The second part follows simply by using the rescaling $\hat{t} := \varphi^{3/2} t$, $\hat{\varphi}_i := \varphi \varphi_i$, and $\hat{X}_i := \varphi X_i(\hat{t})$. □

Remark 3.2. *Using the same type of rescalings as the one leading to (3.2), it is not hard to prove that $T \rightarrow E(T)$ is a continuous function.*

As a consequence of the monotonicity of the support and of the previous lemma, we can derive the following fundamental recursive characterization of $E(T)$.

Lemma 3.3. *For every $T > 0$,*

$$(3.4) \quad E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(T \varphi_i^{-3/2}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2,$$

where \bar{X}_i are the centers of the intervals of length φ_i i.e. $\bar{X}_i = -\frac{1}{2} + \sum_{j < i} \varphi_j + \frac{\varphi_i}{2}$.

Proof. Let $\varphi_1, \dots, \varphi_N$ be the fluxes of the branches leaving from the point $(0, 0)$ (if it is not a branching point then $N = 1$). Up to relabeling, we may assume that the φ_i are ordered i.e. φ_1 corresponds to the first branch, φ_2 to the second and so on. By the monotonicity of the traces (Proposition 2.7), the N branches are independent and the mass from the first branch will go to $[-1/2, -1/2 + \varphi_1]$, the second branch will go to $[-1/2 + \varphi_1, -1/2 + \varphi_1 + \varphi_2]$ and so on. Therefore, from (3.1) and (3.2),

$$E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N E(T, \varphi_i, \bar{X}_i) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(T \varphi_i^{-3/2}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2.$$

□

Before going further, let us point out that for $T, t > 0$ using as test configuration for $E(T + t)$, δ_0 in $[0, t]$ extended by the minimizer of $E(T)$ in $[t, T + t]$, we obtain

$$(3.5) \quad E(T + t) \leq E(T) + t.$$

This together with (3.4), motivates the introduction of the largest branching time:

$$(3.6) \quad T_* = \inf\{T : E(T + t) = E(T) + t, \quad \forall t \geq 0\}.$$

By definition of T_* and (3.5), we see that for every $\varepsilon > 0$, $E(T_* - \varepsilon) + \varepsilon > E(T_*)$ which means that every minimizer of (3.4) for $T = T_*$ must have $N > 1$ branches (there must be branching at time zero).

We will also need the following simple lemma.

Lemma 3.4. *Let $X \in \mathbb{R}$, $T, \varphi > 0$ with $T\varphi^{-3/2} > T_*$. And let μ_t be a minimizer for $E(T, \varphi, X)$. Letting $X(t) := (1 - \frac{t}{T})X$, for $t \in [0, T - \varphi^{3/2}T_*]$ it holds, $\mu_t = \varphi\delta_{X(t)}$.*

Proof. Since $T\varphi^{-3/2} > T_*$, by definition of T_* , in $[0, T\varphi^{-3/2} - T_*]$, every minimizer of $E(T\varphi^{-3/2})$ is of the form δ_0 . Therefore, by (3.1) and (3.2), if $X(t) := (1 - \frac{t}{T})X$, then for $t \in [0, T - \varphi^{3/2}T_*]$, $\mu_t = \varphi\delta_{X(t)}$. \square

We can now state the main result of this section.

Proposition 3.5. *We have $T_* = 1/4$ and if $\varphi_1, \dots, \varphi_N$ are optimal in (3.4) for $T = T_*$, then $N = 2$ and $\varphi_1 = \varphi_2 = 1/2$. Moreover,*

$$E(1/4) - 1/4 = \frac{1}{2 - \sqrt{2}}.$$

The proof of this proposition will consist of the remaining part of this section. Before doing so, let us see how it implies Theorem 1.1. In the proof, we will use the following notation

Definition 3.6. *For $\mu \in \mathcal{A}_{a,b}^*$ with $\mu_t = \sum_i \varphi_i \delta_{X_i}$ and $X \in \mathbb{R}$, let $S_X(\mu)$ be the measure defined by $(S_X(\mu))_t := \sum_i \varphi_i \delta_{X_i + X}$.*

Proof of Theorem 1.1. By definition of T_* , for $T \geq 1/4$, we have $E(T) = E(1/4) + (T - 1/4)$ and if μ is a minimizer for $E(T)$, then it coincides with δ_0 in $[0, T - 1/4]$ and with (a translated version of) a minimizer for $E(1/4)$ in $[T - 1/4, T]$. Therefore, it is enough proving that for $T = 1/4$, the only minimizer of $E(1/4)$ is given by μ^* .

Let μ be such a minimizer and let us prove by induction that $\mu = \mu^*$. Recall first that we defined $t_k = \frac{1}{4} \left(1 - \left(\frac{1}{2}\right)^{3k/2}\right)$. Assume that $\mu_t = \mu_t^*$ for $t \in [0, t_{k-1}]$ and that $\mu_{t_{k-1}} = 2^{-(k-1)} \sum_{i=1}^{2^{k-1}} \delta_{X_i^{k-1}}$ for some ordered $X_i^{k-1} \in \mathbb{R}$. By monotonicity of the support, in $[t_{k-1}, 1/4]$, each of the forward subsystems $\mu^{+,i}$ emanating from X_i^{k-1} must be of the form $\mu_t^{+,i} = S_{\overline{X}_i^{k-1}}(\mu_{t-t_{k-1}}^i)$ where μ^i is a minimizer of $E(\frac{1}{4} - t_{k-1}, 2^{-(k-1)}, X_i^{k-1} - \overline{X}_i^{k-1}) = E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-(k-1)}, X_i^{k-1} - \overline{X}_i^{k-1})$. By (3.2) and Proposition 3.5, every minimizer of

$E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-(k-1)}, X_i^{k-1} - \bar{X}_i^{k-1})$ must branch into two pieces of equal mass. Thus, we can further decompose $\mu^i = \mu^{i,1} + \mu^{i,2}$ where $\mu^{i,1} = S_{-2^{-(k+1)}}(\nu^{i,1})$ with $\nu^{i,1}$ a minimizer for $E(\frac{1}{4}(2^{-(k-1)})^{3/2}, 2^{-k}, 2^{-(k+1)} + X_i^{k-1} - \bar{X}_i^{k-1})$ and similarly for μ_i^2 . Let

$$Y_{2i-1}^k(s) := -2^{-(k+1)} + (1 - \frac{s}{\frac{1}{4} - t_{k-1}})(X_i^{k-1} - \bar{X}_{2i-1}^k) \quad \text{and}$$

$$Y_{2i}^k(s) := 2^{-(k+1)} + (1 - \frac{s}{\frac{1}{4} - t_{k-1}})(X_i^{k-1} - \bar{X}_{2i}^k),$$

Since $\bar{X}_i^{k-1} - 2^{-(k+1)} = \bar{X}_{2i-1}^k$ and $\bar{X}_i^{k-1} + 2^{-(k+1)} = \bar{X}_{2i}^k$, by Lemma 3.4, for $s \in [0, t_k - t_{k-1}]$, $\mu_s^i = 2^{-k}(\delta_{Y_{2i-1}^k(s)} + \delta_{Y_{2i}^k(s)})$ and thus letting

$$X_{2i-1}^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}}(\bar{X}_{2i-1}^k - X_i^{k-1}) + X_i^{k-1} \quad \text{and}$$

$$X_{2i}^k(t) := \frac{t - t_{k-1}}{\frac{1}{4} - t_{k-1}}(\bar{X}_{2i}^k - X_i^{k-1}) + X_i^{k-1},$$

we finally obtain as claimed that for $t \in [t_{k-1}, t_k]$,

$$\mu_t^{+,i} = 2^{-k}(\delta_{X_{2i-1}^k(t)} + \delta_{X_{2i}^k(t)}).$$

□

We may start investigating the properties of T_* .

Lemma 3.7. *It holds*

$$0 < T_* < \infty.$$

As a consequence, the infimum in (3.6) is attained.

Proof. We first observe that for every $T > 0$, by (2.4)

$$(3.7) \quad E(T) \geq T + \frac{W_2^2(\delta_0, dx \llcorner [-1/2, 1/2])}{T}.$$

Let us prove that $T_* < \infty$. Let $T \geq 1$ and μ_t be a minimizer for $E(T)$. By (3.5), for every $T \geq 1$,

$$E(T) \leq E(1) + (T - 1).$$

On the other hand, by the no-loop condition, if μ_t has its first branching at time t_0 then in $[t_0, T]$ it has at least two branches and thus

$$E(T) \geq 2(T - t_0).$$

Putting these two inequalities together we get $t_0 \geq T - (E(1) - 1)$. Letting $T_1 := E(1) - 1$, which is positive by (3.7), and assuming that $T \geq T_1$, this implies that before $T - T_1$, no branching may occur. Hence, for $T \geq T_1$,

$$E(T) = E(T_1) + (T - T_1),$$

that is $T_* \leq T_1$.

We now prove that $T_* > 0$. By (3.7), if $T_* = 0$, for every $T_1 \leq T$,

$$E(T) = E(T_1) + T - T_1 \geq T + \frac{W_2^2(\delta_0, dx \llcorner [-1/2, 1/2])}{T_1}$$

which letting $T_1 \rightarrow 0$ would give a contradiction to $E(T) < \infty$. The fact that the infimum in (3.6) is attained follows by continuity of $T \rightarrow E(T)$. \square

The next result is a form of equipartition of energy.

Lemma 3.8. *If $\varphi_1, \dots, \varphi_N > 0$ are such that*

$$E(T_*) = \sum_{i=1}^N \varphi_i^{3/2} E(T_* \varphi_i^{-3/2}) + \frac{1}{T_*} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2,$$

then

$$(3.8) \quad T_*(N-1) = \frac{1}{T_*} \sum_{i=1}^N \varphi_i |\bar{X}_i|^2.$$

Proof. Let $\mu_t = \sum_i \varphi_i \delta_{X_i}$ be optimal for T_* . By definition of T_* , the fact that $T_* \varphi_i^{-3/2} > T_*$ and Lemma 3.4, there is $\bar{\varepsilon} > 0$ such that for $t \in [0, \bar{\varepsilon}]$, $X_i(t) = \frac{t}{T_*} \bar{X}_i$. For $\bar{\varepsilon} > \varepsilon > 0$, we are going to construct a competitor for $E(T_* - \varepsilon)$. In $[\bar{\varepsilon} - \varepsilon, T_* - \varepsilon]$, let $Y_i(t) := X_i(t + \varepsilon)$ and in $[0, \bar{\varepsilon} - \varepsilon]$, $Y_i(t) := \frac{1}{T_*} \frac{\bar{\varepsilon}}{\bar{\varepsilon} - \varepsilon} t \bar{X}_i$ so that $Y_i(\bar{\varepsilon} - \varepsilon) = X_i(\bar{\varepsilon})$. Therefore,

$$\begin{aligned} E(T_* - \varepsilon) &\leq E(T_*) - N\varepsilon + \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 (\bar{\varepsilon} - \varepsilon) \frac{\bar{\varepsilon}^2}{(\bar{\varepsilon} - \varepsilon)^2} - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \bar{\varepsilon} \\ &= E(T_*) - \left(N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \right) \varepsilon + o(\varepsilon^2) \end{aligned}$$

Hence,

$$\frac{E(T_* - \varepsilon) - E(T_*)}{-\varepsilon} \geq N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 + o(\varepsilon).$$

Using (3.5) and letting $\varepsilon \rightarrow 0$, we get

$$N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \leq 1.$$

We similarly define a competitor for $E(T_* + \varepsilon)$ by letting $Y_i(t) := X_i(t - \bar{\varepsilon})$ in $[\varepsilon + \bar{\varepsilon}, T_* + \varepsilon]$ and $Y_i(t) := \frac{1}{T_*} \frac{\bar{\varepsilon}}{\varepsilon + \bar{\varepsilon}} t \bar{X}_i$ in $[0, \varepsilon + \bar{\varepsilon}]$ and get

$$\begin{aligned} E(T_* + \varepsilon) &\leq E(T_*) + N\varepsilon + \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 (\bar{\varepsilon} + \varepsilon) \frac{\bar{\varepsilon}^2}{(\bar{\varepsilon} + \varepsilon)^2} - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \bar{\varepsilon} \\ &= E(T_*) + \left(N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \right) \varepsilon + o(\varepsilon^2). \end{aligned}$$

From this we infer that

$$\frac{E(T_* + \varepsilon) - E(T_*)}{\varepsilon} \leq N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 + o(\varepsilon).$$

By definition of T_* and by continuity of E ,

$$\frac{E(T_* + \varepsilon) - E(T_*)}{\varepsilon} = 1$$

and therefore

$$N - \frac{1}{T_*^2} \sum_i \varphi_i |\bar{X}_i|^2 \geq 1,$$

which conclude the proof of (3.8). \square

Remark 3.9. Notice that (3.8) is compatible with $N = 2$, $\varphi_1 = \varphi_2 = 1/2$ and $T_* = \frac{1}{4}$.

The following lemma gives an explicit formula for the second term in (3.4).

Lemma 3.10. For every $(\varphi_i)_{i=1}^N$ with $\sum_{i=1}^N \varphi_i = 1$, it holds

$$(3.9) \quad \sum_{i=1}^N \varphi_i |\bar{X}_i|^2 = \frac{1}{12} \left(1 - \sum_{i=1}^N \varphi_i^3 \right).$$

Proof. We prove this by induction on N . For $N = 1$, there is nothing to prove. For $N = 2$, since $\varphi_2 = 1 - \varphi_1$, the left-hand side of (3.9) is equal to

$$\varphi_1 \left(\frac{-1 + \varphi_1}{2} \right)^2 + (1 - \varphi_1) \left(\frac{\varphi_1}{2} \right)^2 = \frac{1}{4} \varphi_1 (1 - \varphi_1),$$

which is equal to the right-hand side of (3.9).

Assume now that (3.9) holds for $N - 1$. Let then $\varphi = \varphi_1 + \varphi_2$ and $\bar{X} = \bar{X}_1 + \frac{\varphi_2}{2}$. By the induction hypothesis,

$$\sum_{i=3}^N \varphi_i |\bar{X}_i|^2 + \varphi |\bar{X}|^2 = \frac{1}{12} \left(1 - \sum_{i=3}^N \varphi_i^3 - \varphi^3 \right)$$

so that

$$\sum_{i=1}^N \varphi_i |\bar{X}_i|^2 = \frac{1}{12} \left(1 - \sum_{i=1}^N \varphi_i^3 \right) + \frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) + \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_2|^2 - \varphi |\bar{X}|^2.$$

We are thus left to prove that

$$(3.10) \quad \frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) + \varphi_1 |\bar{X}_1|^2 + \varphi_2 |\bar{X}_2|^2 - \varphi |\bar{X}|^2 = 0.$$

By definition of φ , \bar{X} and since $\bar{X}_2 = \bar{X}_1 + \frac{\varphi_1 + \varphi_2}{2}$, we have

$$\frac{1}{12} (\varphi_1^3 + \varphi_2^3 - \varphi^3) = -\frac{\varphi_1 \varphi_2}{4} (\varphi_1 + \varphi_2) = -\frac{\varphi_1 \varphi_2 \varphi}{4}.$$

Analogously we can compute

$$\begin{aligned}
\varphi_1|\overline{X}_1|^2 + \varphi_2|\overline{X}_2|^2 - \varphi|\overline{X}|^2 &= \varphi_1|\overline{X}_1|^2 + \varphi_2|\overline{X}_1 + \frac{\varphi}{2}|^2 - \varphi|\overline{X}_1 + \frac{\varphi_2}{2}|^2 \\
&= \varphi_1|\overline{X}_1|^2 + \varphi_2|\overline{X}_1|^2 + \varphi_2\varphi\overline{X}_1 + \frac{\varphi_2\varphi^2}{4} - \varphi|\overline{X}_1|^2 - \varphi\varphi_2\overline{X}_1 - \frac{\varphi\varphi_2^2}{4} \\
&= \frac{\varphi_2\varphi}{4}(\varphi - \varphi_2) \\
&= \frac{\varphi_1\varphi_2\varphi}{4}.
\end{aligned}$$

Adding these two equalities we get (3.10) which concludes the proof of (3.9). \square

As a consequence, we obtain from (3.4) that for every $T > 0$,

$$(3.11) \quad E(T) = \min_{\sum_{i=1}^N \varphi_i = 1} \sum_{i=1}^N \varphi_i^{3/2} E(\varphi_i^{-3/2} T) + \frac{1}{12T} \left(1 - \sum_{i=1}^N \varphi_i^3 \right).$$

In this form, it is clear that the energy does not change if we reorder the φ_i .

Using the characterization (3.11), we show another characterization of $E(T_*)$ which has the advantage of not being recursive anymore.

Proposition 3.11. *It holds*

$$(3.12) \quad \frac{E(T_*) - T_*}{T_*} = \min_{\sum_{i=1}^N \varphi_i = 1} \frac{(N-1) + \frac{1}{12T_*^2} \left(1 - \sum_{i=1}^N \varphi_i^3 \right)}{1 - \sum_{i=1}^N \varphi_i^{3/2}}.$$

Moreover, if $\varphi_1, \dots, \varphi_N$ are minimizers for $E(T_*)$ then they also minimize the right-hand side of (3.12) and vice-versa.

Proof. Let $\overline{\varphi}_i$ be the optimal fluxes for (3.11). By definition of T_* , we have for every φ_i with $\sum_i \varphi_i = 1$ (since $\varphi_i^{-3/2} \geq 1$)

$$\begin{aligned}
E(T_*) &\leq \sum_{i=1}^N \varphi_i^{3/2} E(T_* \varphi_i^{-3/2}) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3 \right) \\
&= \sum_{i=1}^N \varphi_i^{3/2} (E(T_*) + (T_* - T_* \varphi_i^{-3/2})) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3 \right) \\
&= E(T_*) \left(\sum_{i=1}^N \varphi_i^{3/2} \right) + T_* \sum_{i=1}^N (1 - \varphi_i^{3/2}) + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3 \right).
\end{aligned}$$

Therefore

$$E(T_*) - T_* \leq (E(T_*) - T_*) \left(\sum_{i=1}^N \varphi_i^{3/2} \right) + (N-1)T_* + \frac{1}{12T_*} \left(1 - \sum_{i=1}^N \varphi_i^3 \right),$$

and then

$$\frac{E(T_*) - T_*}{T_*} \leq \frac{(N-1) + \frac{1}{12T_*^2} \left(1 - \sum_{i=1}^N \varphi_i^3\right)}{1 - \sum_{i=1}^N \varphi_i^{3/2}}$$

with equality for $\varphi_i = \bar{\varphi}_i$. □

For $N \geq 2$, we introduce a quantity which will play a central role in our analysis. Let

$$\alpha_N := \inf_{\varphi_i \geq 0} \left\{ \frac{1 - \sum_{i=1}^N \varphi_i^3}{1 - \sum_{i=1}^N \varphi_i^{3/2}} : \sum_{i=1}^N \varphi_i = 1 \right\}.$$

Proposition 3.12. *It holds*

$$T_* \leq \frac{1}{4}.$$

Moreover, the number $N > 1$ of branches of the minimizer for T_* , satisfies

$$(3.13) \quad \sqrt{N}(\sqrt{N} + 1)T_* + \frac{\alpha_N}{12T_*} \leq \frac{\sqrt{2}}{2(\sqrt{2} - 1)}.$$

As a consequence,

$$(3.14) \quad \sqrt{N} \leq \frac{1}{2} \left(-1 + \left(1 + \frac{6}{\alpha_N(\sqrt{2} - 1)^2} \right)^{1/2} \right).$$

In particular, since $\alpha_N \geq 1$, this gives $N \leq 6$.

Proof. We start with the upper bound $T_* \leq \frac{1}{4}$. Let $\varphi_1, \dots, \varphi_N$ be optimal for $E(T_*)$. Then, by (3.12) and (3.8),

$$\frac{E(T_*) - T_*}{T_*} = \frac{2(N-1)}{\left(1 - \sum_i \varphi_i^{3/2}\right)}.$$

Using that for every $\varphi_1, \dots, \varphi_N$ with $\sum \varphi_i = 1$,

$$(3.15) \quad \frac{1}{1 - \sum_i \varphi_i^{3/2}} \geq \frac{1}{1 - \sqrt{N}^{-1}}$$

we get

$$\frac{E(T_*) - T_*}{T_*} \geq \frac{2(N-1)\sqrt{N}}{\sqrt{N} - 1}.$$

Since the right-hand side is minimized for $N = 2$ (among $N \in \mathbb{N}$, $N \geq 2$), we have

$$(3.16) \quad \frac{E(T_*) - T_*}{T_*} \geq \frac{2\sqrt{2}}{\sqrt{2} - 1}.$$

We proceed further by proving an upper bound for the left-hand side of (3.16). For every $T > 0$, we can construct the self similar competitor for which every branch is divided into

two branches of half the mass at every branching point (which is at $T_k = T(1 - (\frac{1}{2})^{3k/2})$). Let $\tilde{E}(T)$ be its energy. Then, arguing as in (3.4), we have

$$\tilde{E}(T) = 2 \left(\frac{1}{2} \right)^{3/2} \tilde{E}(T) + 2 \left(T - \left(\frac{1}{2} \right)^{3/2} T \right) + \frac{1}{16T}$$

that is

$$(\tilde{E}(T) - T) \left(1 - \frac{1}{\sqrt{2}} \right) = T + \frac{1}{16T}$$

from which we get

$$\frac{\tilde{E}(T) - T}{T} = \frac{\sqrt{2}}{\sqrt{2} - 1} \left(1 + \frac{1}{16T^2} \right).$$

Since by definition $E(T) \leq \tilde{E}(T)$, we get

$$(3.17) \quad \frac{E(T) - T}{T} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \left(1 + \frac{1}{16T^2} \right).$$

For $T > \frac{1}{4}$, the right-hand side is strictly smaller than $\frac{2\sqrt{2}}{\sqrt{2}-1}$ hence by (3.16), we cannot have $T = T_*$. This gives the upper bound.

We now turn to (3.13). For this, we notice that

$$(3.18) \quad \frac{E(T_*) - T_*}{T_*} = \frac{N - 1}{1 - \sum_i \varphi_i^{3/2}} + \frac{(1 - \sum_i \varphi_i^3)}{12T_*^2(1 - \sum_i \varphi_i^{3/2})} \geq \frac{N - 1}{1 - N^{-1/2}} + \frac{\alpha_N}{12T_*^2}.$$

Since $T_* \leq 1/4$, by definition of T_* (recall (3.6)), $E(1/4) - 1/4 = E(T_*) - T_*$, combining (3.18) with (3.17) for $T = 1/4$ and

$$\frac{N - 1}{1 - N^{-1/2}} = \sqrt{N}(\sqrt{N} + 1),$$

yields (3.13). We finally derive (3.14). For this, multiply (3.13) by T_* to obtain that

$$\sqrt{N}(\sqrt{N} + 1)T_*^2 - \frac{\sqrt{2}}{2(\sqrt{2} - 1)}T_* + \frac{\alpha_N}{12} \leq 0.$$

This implies that the polynomial $\sqrt{N}(\sqrt{N} + 1)X^2 - \frac{\sqrt{2}}{2(\sqrt{2}-1)}X + \frac{\alpha_N}{12}$ has real roots (and that T_* lies between these two roots) so that

$$\Delta = \frac{1}{2(\sqrt{2} - 1)^2} - \frac{\alpha_N}{3} \sqrt{N}(\sqrt{N} + 1) \geq 0,$$

which is equivalent to

$$(\sqrt{N})^2 + \sqrt{N} - \frac{3}{2\alpha_N(\sqrt{2} - 1)^2} \leq 0.$$

Since the largest root of this polynomial (in the variable \sqrt{N}) is given by $\frac{1}{2}(-1 + (1 + \frac{6}{\alpha_N(\sqrt{2}-1)^2})^{1/2})$, we have obtained (3.14). \square

N	3	4	5	6
$\alpha_N >$	1.76	1.4	1.17	1.01

TABLE 1. Critical values of α_N .

Remark 3.13. *From the proof of the previous proposition, one could also get a lower bound for T_* . We will make use of this fact later on to study the case $N = 2$.*

3.1. Excluding the case $N > 2$. In this section, we show by a computer assisted proof that for $T = T_*$, there cannot be more than two branches.

Lemma 3.14. *For $3 \leq N \leq 6$,*

$$(3.19) \quad \alpha_N > \frac{6}{(\sqrt{2} - 1)^2(2\sqrt{N} + 1)^2}.$$

As a consequence, the number of branches at $T = T_$ equals two.*

Proof. By inverting the relation between α_N and N in (3.14), it is readily seen that (3.19) and (3.14) are incompatible. Therefore, proving the lower bound (3.19) directly excludes the possibility of having $N > 2$ branches.

In Table 1, the values of the right-hand side of (3.19) are given for $N \in [3, 6]$. Since we are not able to prove these bounds analytically, we resort to a computer assisted proof. From Table 1, we see that we want to compute α_N with a precision of 10^{-2} .

Step 1 (Computation of α_2):

We start by computing α_2 . We claim that

$$(3.20) \quad \alpha_2 = 2.$$

Since

$$\alpha_2 = \min_{\varphi \in [0, 1/2]} \frac{3\varphi(1 - \varphi)}{1 - \varphi^{3/2} - (1 - \varphi)^{3/2}},$$

in order to prove that the minimum is attained for $\varphi = 0$, it is enough to show that for $\varphi \in [0, 1/2]$,

$$\varphi(1 - \varphi) \geq \frac{2}{3}(1 - \varphi^{3/2} - (1 - \varphi)^{3/2}).$$

Since the two expressions agree for $\varphi = 0$, by differentiating, we are left with the proof of

$$1 - 2\varphi \geq (1 - \varphi)^{1/2} - \varphi^{1/2}$$

or equivalently of

$$1 - 2\varphi + \varphi^{1/2} \geq (1 - \varphi)^{1/2}.$$

Squaring both sides (notice that $1 - 2\varphi + \varphi^{1/2} \geq 0$), this amounts to prove that for $\varphi \in [0, 1/2]$,

$$(3.21) \quad 2\varphi^2 - 2\varphi^{3/2} - \varphi + \varphi^{1/2} \geq 0.$$

Since the polynomial,

$$P(X) = 2X^4 - 2X^3 - X^2 + X$$

has roots $\{-1/\sqrt{2}, 0, 1/\sqrt{2}, 1\}$, it is positive in $[0, 1/\sqrt{2}]$ and thus considering $X = \varphi^{1/2}$, we see that (3.21) holds and thus (3.20) is proven.

Step 2 (Lower bound for α_N): Consider now $N \geq 3$. Since $\frac{1-\sum_i \varphi_i^3}{1-\sum_i \varphi_i^{3/2}}$ is continuous on the compact convex set $K = \{0 \leq \varphi_i \leq 1 : \sum_i \varphi_i = 1\}$ (the only problem could arise when all the φ_i 's but one go to zero but then it is easy to see that $\frac{1-\sum_i \varphi_i^3}{1-\sum_i \varphi_i^{3/2}} \rightarrow 2$), the minimum is attained. If the minimum is attained at the boundary then $\alpha_N = \alpha_{N-1}$. Since the values in Table 1 are decreasing with N and since $\alpha_2 = 2$, in that case by a simple induction we would be over. Otherwise, we claim that for every N , the optimal φ_i may take only two values: φ repeated $k \in [1, N]$ times and $\frac{1-k\varphi}{N-k}$ repeated $N-k$ times. Indeed, fix $i \neq j \in [1, N]$ and for $|\varepsilon|$ small enough, define $\hat{\varphi}_i := \varphi_i + \varepsilon$, $\hat{\varphi}_j := \varphi_j - \varepsilon$ and $\hat{\varphi}_k := \varphi_k$ for $k \neq i, j$. By minimality, the derivative in zero of

$$\varepsilon \rightarrow \frac{1 - \sum_k \hat{\varphi}_k^3}{1 - \sum_k \hat{\varphi}_k^{3/2}}$$

is equal to zero. From this, we get the condition

$$\varphi_i^2 - \frac{\alpha_N}{2} \varphi_i^{1/2} = \varphi_j^2 - \frac{\alpha_N}{2} \varphi_j^{1/2} \quad \forall i \neq j.$$

Letting $P_N(X) := X^4 - \frac{\alpha_N}{2} X$, this means that for every $i \neq j$, $P_N(\varphi_i^2) = P_N(\varphi_j^2)$. Since P'_N has only one positive root, this means that φ_i may take at most two values, proving the claim.

Therefore,

$$(3.22) \quad \alpha_N = \min_{k \in [1, N]} \min_{\varphi \in [0, k^{-1}]} f_{k, N}(\varphi),$$

where

$$f_{k, N}(\varphi) := \frac{1 - k\varphi^3 - (N-k)^{-2}(1 - k\varphi)^3}{1 - k\varphi^{3/2} - (N-k)^{-1/2}(1 - k\varphi)^{3/2}}.$$

We are thus left to estimate a finite number of one dimensional functions. Let us first point out that the case $k = N$ corresponds to $\varphi = 1/N$. In this case, a simple numerical check shows that $f_{k, N}(1/N)$ is larger than the critical values computed in Table 1.

For $3 \leq N \leq 6$ and $1 \leq k \leq N-1$, we want to estimate $\min_{\varphi \in [0, k^{-1}]} f_{k, N}(\varphi)$ with a precision of 10^{-2} . For this we will compute the values of $f_{k, N}$ for a sufficiently fine discretization of $[0, k^{-1}]$. Let $I \subseteq [0, 1/k]$ and let $\Lambda := \sup_{\varphi \in I} |f'_{k, N}(\varphi)|$. Since for $\varphi, \psi \in I$,

$$|f_{k, N}(\varphi) - f_{k, N}(\psi)| \leq \Lambda |\varphi - \psi|,$$

in order to get a precision of 10^{-2} on $\inf_I f_{k,N}$, it is enough to use a discretization step $\delta \leq 10^{-2}\Lambda^{-1}$. We are thus naturally led to estimate $\sup |f'_{k,N}|$. We can compute

$$(3.23) \quad f'_{k,N}(\varphi) = \frac{3k}{1 - k\varphi^{3/2} - (N-k)^{-1/2}(1 - k\varphi)^{3/2}} \left(-\varphi^2 + (N-k)^{-2}(1 - k\varphi)^2 - \frac{1}{2}(-\varphi^{1/2} + (N-k)^{-1/2}(1 - k\varphi)^{1/2})f_{k,N}(\varphi) \right).$$

Step 2.1 (the case $k \in [2, N-2]$): We start with the easier case $k \in [2, N-2]$ (which in particular implies $N \geq 4$). In this case, $\sup_{[0,1/k]} |f'_{k,N}| < \infty$ and we can take $I = [0, 1/k]$. Since it will appear in several places, we first study the function

$$g_{k,N}(\varphi) := 1 - k\varphi^{3/2} - (N-k)^{-1/2}(1 - k\varphi)^{3/2}.$$

In particular, we want to study $\min g_{k,N}$. Taking the derivative, we obtain

$$g'_{k,N}(\varphi) = \frac{3k}{2}(-\varphi^{1/2} + (N-k)^{-1/2}(1 - k\varphi)^{1/2}).$$

Therefore, $g'_{k,N}$ is zero only if $\varphi = 1/N$ so that $g'_{k,N}$ is first positive and then negative and $g_{k,N}$ attains its minimum on the boundary. Hence, for $k \in [2, N-2]$,

$$\min_{\varphi \in [0, 1/k]} g_{k,N}(\varphi) = \min(1 - k^{-1/2}, 1 - (N-k)^{-1/2}) \geq 1 - 2^{-1/2}.$$

Injecting this into (3.23), we find

$$\begin{aligned} |f'_{k,N}(\varphi)| &\leq \frac{3k}{1 - 2^{-1/2}} \left(k^{-2} + (N-k)^{-2} + \frac{1}{2}(1 - 2^{-1/2})^{-1}(k^{-1/2} + (N-k)^{-1/2}) \right) \\ &\leq \frac{6}{1 - 2^{-1/2}} \left(1 + \frac{2}{\sqrt{2} - 1} \right) \\ &\leq 120. \end{aligned}$$

In this case, we may take

$$(3.24) \quad \delta \leq 12^{-1} \times 10^{-3}.$$

Step 2.2 (the case $k \in \{1, N-1\}$): We turn to the more delicate case of $k = N-1$ (the case $k = 1$ being analogous). In this case, $\sup_{\varphi \in [0, 1/k]} |f'_{k,N}(\varphi)| = \lim_{\varphi \rightarrow 0} |f'_{k,N}(\varphi)| = \infty$. Therefore, we need to be more careful. A Taylor expansion shows that

$$\lim_{\varphi \rightarrow 0} f_{k,N}(\varphi) = 2,$$

which is always strictly bigger than the critical values computed in Table 1. Hence, if we can find $\eta > 0$, such that in $[0, \eta]$, $f'_{k,N}$ is positive, we will have

$$\alpha_N = \min \left(\min_{\varphi \in [0, \eta]} f_{k,N}(\varphi), \min_{\varphi \in [\eta, 1/k]} f_{k,N}(\varphi) \right) = \min(2, \min_{\varphi \in [\eta, 1/k]} f_{k,N}(\varphi)).$$

From (3.23), we see that $f'_{N-1,N}$ is of the same sign as

$$h_N(\varphi) := -\varphi^2 + (1 - (N-1)\varphi)^2 + \frac{1}{2}(\varphi^{1/2} - (1 - (N-1)\varphi)^{1/2})f_{N-1,N}(\varphi).$$

Since for every $\varphi \geq 0$,

$$-\varphi^2 + (1 - (N-1)\varphi)^2 \geq 1 - 2(N-1)\varphi,$$

and

$$-(1 - (N-1)\varphi)^{1/2} \geq -1,$$

we can first infer that

$$(3.25) \quad h_N(\varphi) \geq 1 - 2(N-1)\varphi + \frac{1}{2}(\varphi^{1/2} - 1)f_{N-1,N}(\varphi).$$

We may now bound from above $f_{N-1,N}(\varphi)$. Notice that regarding the numerator,

$$(3.26) \quad 1 - (N-1)\varphi^3 - (1 - (N-1)\varphi)^3 \leq 3(N-1)\varphi + (N-1)^3\varphi^3.$$

In order to bound from below the denominator, we first claim that for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$,

$$(3.27) \quad (1 - (N-1)\varphi)^{3/2} \leq 1 - \frac{3}{2}(N-1)\varphi + \frac{(N-1)}{4}\varphi^{3/2}.$$

Indeed, letting

$$\Psi_N(\varphi) := (1 - (N-1)\varphi)^{3/2} - 1 + \frac{3}{2}(N-1)\varphi - \frac{(N-1)}{4}\varphi^{3/2},$$

we have $\Psi_N(0) = 0$ and

$$\Psi'_N(\varphi) = \frac{3}{2}(N-1) \left(1 - \frac{1}{4}\varphi^{1/2} - (1 - (N-1)\varphi)^{1/2}\right).$$

Since for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$, $1 - \frac{1}{4}\varphi^{1/2} - (1 - (N-1)\varphi)^{1/2} \leq 0$, we have $\Psi'_N \leq 0$ and thus also $\Psi_N \leq 0$, proving (3.27). From this we get that for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$,

$$(3.28) \quad 1 - (N-1)\varphi^{3/2} - (1 - (N-1)\varphi)^{3/2} \geq \frac{3}{2}(N-1)\varphi - \frac{5}{4}(N-1)\varphi^{3/2}.$$

Putting (3.26) and (3.28) together, we get that for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$,

$$(3.29) \quad f_{N-1,N}(\varphi) \leq 2 \frac{1 + \frac{(N-1)^2}{3}\varphi^2}{1 - \frac{5}{6}\varphi^{1/2}}.$$

Inserting this back into (3.25), we get that for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$,

$$h_N(\varphi) \geq 1 - 2(N-1)\varphi + (\varphi^{1/2} - 1) \frac{1 + \frac{(N-1)^2}{3}\varphi^2}{1 - \frac{5}{6}\varphi^{1/2}}.$$

Since for $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$, $5/6\varphi^{1/2} \leq 2/3$ and thus $\frac{\varphi^2(N-1)^2}{3(1-\frac{5}{6}\varphi^{1/2})} \leq \varphi^2(N-1)^2 \leq \varphi(N-1)$, this may be further simplified into

$$h_N(\varphi) \geq 1 - 3(N-1)\varphi + (\varphi^{1/2} - 1)\frac{1}{1 - \frac{5}{6}\varphi^{1/2}}.$$

Since for $x \leq 1/7$,

$$\frac{1}{1-x} \leq 1 + \frac{7}{6}x,$$

and since $\varphi \leq \left(\frac{8}{1+16(N-1)}\right)^2$ implies $5/6\varphi^{1/2} \leq 1/7$, we have

$$\begin{aligned} h_N(\varphi) &\geq 1 - 3(N-1)\varphi + (\varphi^{1/2} - 1)\left(1 + \frac{35}{36}\varphi^{1/2}\right) \\ &\geq \frac{1}{36}\varphi^{1/2} - 3(N-1)\varphi. \end{aligned}$$

In particular, if $\varphi \leq \frac{1}{108(N-1)}$, then $h_N \geq 0$. To sum up, we have proven that

$$f'_{N-1,N} \geq 0 \quad \text{in} \quad \left[0, \min\left(\left(\frac{8}{1+16(N-1)}\right)^2, \frac{1}{108(N-1)}\right)\right].$$

Letting $\eta := \frac{1}{540}$ (which corresponds to $\frac{1}{108(N-1)}$ with $N = 6$), then for $N \in [3, 6]$, $\eta \leq \min\left(\left(\frac{8}{1+16(N-1)}\right)^2, \frac{1}{108(N-1)}\right)$, so that $f'_{N-1,N} \geq 0$ in $[0, \eta]$.

We finally estimate $|f'_{N-1,N}|$ in $[\eta, (N-1)^{-1}]$. Recall that by (3.23)

$$\begin{aligned} f'_{N-1,N}(\varphi) &= \frac{3(N-1)}{g_{N-1,N}(\varphi)} \left(-\varphi^2 + (1 - (N-1)\varphi)^2 \right. \\ &\quad \left. - \frac{1}{2}(-\varphi^{1/2} + (1 - (N-1)\varphi)^{1/2}) \frac{1 - (N-1)\varphi^3 - (1 - (N-1)\varphi)^3}{g_{N-1,N}(\varphi)} \right). \end{aligned}$$

As before,

$$\begin{aligned} \min_{[\eta, (N-1)^{-1}]} g_{N-1,N}(\varphi) &= \min(g_{N-1,N}(\eta), g_{N-1,N}(1/(N-1))) \\ &= \min\left(1 - (N-1)\eta^{3/2} - (1 - (N-1)\eta)^{3/2}, 1 - (N-1)^{-1/2}\right) \\ &\geq 1 - 5\eta^{3/2} - (1 - 5\eta)^{3/2} \geq 1, 3 \times 10^{-2}. \end{aligned}$$

We can thus estimate for $\varphi \in [\eta, (N-1)^{-1}]$,

$$|f'_{N-1,N}(\varphi)| \leq \frac{18 \times 10^2}{1, 3} \left(2 + \frac{10^2}{1, 3}\right) \leq 2 \times 10^5.$$

N	3	4	5	6
$\alpha_N \simeq$	2	1.88	1.74	1.64

TABLE 2. Numerical values of α_N

Hence, we may finally take

$$(3.30) \quad \delta \leq 0.5 \times 10^{-7}.$$

Using a simple Scilab code, we find the values of α_N given in Table 2. We see that the values we find are well above the critical values given in Table 1. \square

Remark 3.15. *Although we are not able to prove it, the numerics show that for $N \geq 4$, the minimum in α_N is attained for equidistributed masses i.e. $\varphi = 1/N$.*

Remark 3.16. *Arguing as in Step 2, it could have been proven that also for the original minimization problem $E(T)$, the optimal masses may take at most two distinct values.*

3.2. Conclusion in the case $N = 2$. In this section, we end the proof of Proposition 3.5 by studying the case $N = 2$.

Let first

$$T_- := \frac{1}{4} \left(1 - \left(1 - \frac{4}{3} (2 - \sqrt{2})^{1/2} \right) \right).$$

Then $1/4 \geq T_* \geq T_-$. Indeed, (3.13) for $N = 2$ (recalling that $\alpha_2 = 2$) may be seen to be equivalent to

$$(3.31) \quad T_*^2 - \frac{1}{2}T_* + \frac{1}{6\sqrt{2}(\sqrt{2}+1)} \leq 0.$$

In particular, T_* has to lie between the two roots of the right-hand side of (3.31) which yields $T_* \geq T_-$ (recall that the bound $1/4 \geq T_*$ was already derived in Proposition 3.12).

Now, if $T_* \leq 1/4$ then $E(1/4) = E(T_*) + 1/4 - T_*$ so that using (3.17) for $T = 1/4$, we get

$$E(T_*) - T_* = E(1/4) - 1/4 \leq \frac{\sqrt{2}}{2(\sqrt{2} - 1)}.$$

Using (3.12), we obtain for φ optimal for T_* , the bound

$$(3.32) \quad T_* \frac{1 + \frac{1}{4T_*^2} \varphi(1 - \varphi)}{1 - \varphi^{3/2} - (1 - \varphi^{3/2})} \leq \frac{\sqrt{2}}{2(\sqrt{2} - 1)}.$$

The next proposition shows the reverse inequality which concludes the proof of Proposition 3.5.

Proposition 3.17. For $T \in [T_-, 1/4]$ and $\varphi \in [0, 1]$,

$$(3.33) \quad T \frac{1 + \frac{1}{4T^2}\varphi(1-\varphi)}{1 - \varphi^{3/2} - (1-\varphi)^{3/2}} \geq \frac{\sqrt{2}}{2(\sqrt{2}-1)},$$

with equality if and only if $T_* = \frac{1}{4}$ and $\varphi = 1/2$.

Proof. By symmetry we can assume that $\varphi \in [0, 1/2]$. Letting $\lambda := 2T$, (3.33) is equivalent to show that for $\lambda \in [2T_-, 1/2]$ and $\varphi \in [0, 1/2]$,

$$(3.34) \quad 1 + \frac{1}{\lambda^2}\varphi(1-\varphi) \geq \frac{\sqrt{2}}{\lambda(\sqrt{2}-1)}(1 - \varphi^{3/2} - (1-\varphi)^{3/2}).$$

It will be more convenient to work with $a := \frac{3\sqrt{2}\lambda}{2(\sqrt{2}-1)}$. Letting

$$a_- := \frac{3\sqrt{2}}{\sqrt{2}-1}T_- = \frac{3}{2(2-\sqrt{2})}(1 - (1 - \frac{4}{3}(2-\sqrt{2}))^{1/2}) \simeq 1.36,$$

we are reduced to $a \in [a_-, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$. Inequality (3.34) then reads

$$(3.35) \quad L(a, \varphi) := 1 + \frac{9}{2a^2(\sqrt{2}-1)^2}\varphi(1-\varphi) \geq \frac{3}{a(\sqrt{2}-1)^2}(1 - \varphi^{3/2} - (1-\varphi)^{3/2}) =: R(a, \varphi).$$

Let us first notice that $L(a, 0) = 1 > 0 = R(a, 0)$ and that for $\varphi = \frac{1}{2}$, (3.35) reads,

$$1 + \frac{9}{8a^2(\sqrt{2}-1)^2} \geq \frac{3}{a\sqrt{2}(\sqrt{2}-1)},$$

which always holds true (in terms of λ , this amounts to $1 + \frac{1}{4\lambda^2} \geq \frac{1}{\lambda}$). Moreover, the inequality above is strict if $a < \frac{3\sqrt{2}}{4(\sqrt{2}-1)}$. We are going to study the variations (for fixed a) of $L(a, \varphi) - R(a, \varphi)$. By differentiating, this is equivalent to study the sign of

$$(3.36) \quad D(\varphi) := 1 - 2\varphi - a((1-\varphi)^{1/2} - \varphi^{1/2}).$$

Let $X := \varphi^{1/2}$. For $a \in [a_-, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$ and $X \in [0, 1/\sqrt{2}]$, since $1 - 2X^2 + aX \geq 0$, the sign of (3.36) is the same as the sign of

$$P(X) := (1 - 2X^2 + aX)^2 - a^2(1 - X^2) = 4X^4 - 4aX^3 + 2(a^2 - 2)X^2 + 2aX + (1 - a^2).$$

Since P has roots $\{\pm 1/\sqrt{2}\}$, we can factor it to obtain

$$P(X) = 2(X^2 - \frac{1}{2})(2X^2 - 2aX + (a^2 - 1)).$$

For $a > \sqrt{2}$, $2X^2 - 2aX + (a^2 - 1)$ has no real roots and therefore, for $a \in [\sqrt{2}, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$, P is negative inside $[0, 1/\sqrt{2}]$ and thus $\partial_\varphi L - \partial_\varphi R \leq 0$ implying that

$$(3.37) \quad \min_{\varphi \in [0, 1/2]} L(a, \varphi) - R(a, \varphi) = L(a, 1/2) - R(a, 1/2) \geq 0,$$

with strict inequality if $a < \frac{3\sqrt{2}}{4(\sqrt{2}-1)}$ or $\varphi \neq 1/2$. This proves (3.34) for $a \in [\sqrt{2}, \frac{3\sqrt{2}}{4(\sqrt{2}-1)}]$. If now $a \in [a_-, \sqrt{2}]$, besides $\pm 1/\sqrt{2}$, P has two more roots

$$(3.38) \quad X_{\pm} := \frac{a \pm \sqrt{2-a^2}}{2}.$$

For $a \in [a_-, \sqrt{2}]$,

$$0 \leq X_- \leq 1/\sqrt{2} \leq X_+,$$

and thus P is negative in $[0, X_-]$ and positive in $[X_-, 1/\sqrt{2}]$ from which,

$$(3.39) \quad \Psi(a) := \min_{\varphi \in [0, 1/2]} L(a, \varphi) - R(a, \varphi) = L(a, X_-^2) - R(a, X_-^2).$$

Let us now prove that for $a \in [a_-, \sqrt{2}]$, $\Psi'(a) \leq 0$. We first compute

$$\Psi'(a) = \partial_a L(a, X_-^2) - \partial_a R(a, X_-^2) + 2X_- \partial_a X_- (\partial_{\varphi} L(a, X_-^2) - \partial_{\varphi} R(a, X_-^2)).$$

By minimality of X_- , $\partial_{\varphi} L(a, X_-^2) - \partial_{\varphi} R(a, X_-^2) = 0$ so that

$$\Psi'(a) = \partial_a L(a, X_-^2) - \partial_a R(a, X_-^2) = \frac{3}{a^2(\sqrt{2}-1)^2} \left(1 - X_-^3 - (1 - X_-^2)^{3/2} - \frac{3}{a} X_-^2 (1 - X_-^2) \right).$$

A simple computation shows that $X_-^2 = \frac{1}{2}(1 - 2a\sqrt{2-a^2})$ so that $\Psi' \leq 0$ is equivalent to

$$\frac{1}{2\sqrt{2}}(1 - 2a\sqrt{2-a^2})^{3/2} + \frac{1}{2\sqrt{2}}(1 + 2a\sqrt{2-a^2})^{3/2} + \frac{3}{4a}(1 - a^2(2-a^2)) \geq 1.$$

This indeed holds since for $a \in [a_-, \sqrt{2}]$,

$$\begin{aligned} \frac{1}{2\sqrt{2}}(1 - 2a\sqrt{2-a^2})^{3/2} + \frac{1}{2\sqrt{2}}(1 + 2a\sqrt{2-a^2})^{3/2} + \frac{3}{4a}(1 - a^2(2-a^2)) &\geq \\ \frac{1}{2\sqrt{2}}((1 - 2a_-\sqrt{2-a_-^2})^{3/2} + 1) + \frac{3}{4\sqrt{2}}(1 - a_-^2(2-a_-^2)) &\simeq 1.02 > 1. \end{aligned}$$

Therefore, $\Psi' \leq 0$ and thus for $a \in [a_-, \sqrt{2}]$, by (3.37)

$$\Psi(a) \geq \Psi(\sqrt{2}) > 0,$$

which ends the proof of (3.33). □

Remark 3.18. From (3.31), one could infer the simpler bound $T_* \geq \frac{1}{3\sqrt{2}(\sqrt{2}+1)}$ which leads to $a \geq 1$. For $a \in [1, \sqrt{2}]$, we still have (3.38) and (3.39). Numerically, it seems that Ψ is decreasing not only in $[a_-, \sqrt{2}]$ but actually on the whole $[1, \sqrt{2}]$. We were unfortunately not able to prove this fact which would have yield a more elegant proof of (3.33).

4. APPLICATIONS AND OPEN PROBLEMS

In this section we use Theorem 1.1 to characterize the symmetric minimizers of

$$(4.1) \quad \min\{\mathcal{E}(\mu) : \mu_{\pm T} = \varphi/Ldx \llcorner [-L/2, L/2]\},$$

at least for T large enough. By rescaling, it is enough to consider $\varphi = L = 1$.

Theorem 4.1. *For $T \in [\frac{1}{4}, \frac{1}{4(2\sqrt{2}-2)})$, the unique symmetric minimizer of (4.1) is equal in $[0, T]$ to $S_{-1/4}(\mu^1) + S_{1/4}(\mu^1)$, where μ^1 is equal to the unique minimizer of $E(T, 1/2)$ given by Corollary 1.2 in $[0, T]$ and to its symmetric in $[-T, 0]$. For $T > 4(2\sqrt{2} - 2)$, it is given by the unique minimizer of $E(T)$ given by Theorem 1.1 in $[0, T]$ and to its symmetric in $[-T, 0]$.*

Proof. By Proposition 2.7, we know that a symmetric minimizer exists. Let μ be such a minimizer. Thanks to the symmetry, we can restrict ourselves to study its structure in $[0, T]$. We let

$$\mathcal{E}^+(\mu) := \int_0^T \#\{\text{supp } \mu_t\} + \sum_i \varphi_i |\dot{X}_i|^2 dt.$$

Let $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$. We first claim that $X_i = \bar{X}_i$, where as before, $\bar{X}_i = \frac{-1}{2} + \sum_{j < i} \varphi_j + \frac{\varphi_i}{2}$. Indeed, applying the same shear as in (3.1), we obtain by minimality of μ ,

$$\mathcal{E}^+(\mu) \geq \mathcal{E}^+(\hat{\mu}) + \frac{1}{T} \sum_{i=1}^N \varphi_i |X_i - \bar{X}_i|^2 \geq \mathcal{E}^+(\mu) + \frac{1}{T} \sum_{i=1}^N \varphi_i |X_i - \bar{X}_i|^2,$$

where the inequality could arise from a decreasing of the number of branches after the shear. This proves the claim. For $i = 1, \dots, N$, let $\mu^{+,i}$ be the forward system emanating from $(X_i, 0)$. Then, by monotonicity of the traces (Proposition 2.7), $\mu_T^{+,i} = dx \llcorner [\bar{X}_i - \varphi_i/2, \bar{X}_i + \varphi_i/2]$ and thus, by the no-loop property

$$(4.2) \quad \mathcal{E}^+(\mu) = \sum_{i=1}^N \mathcal{E}^+(\mu^{+,i}) = \sum_{i=1}^N E(T, \varphi_i).$$

Moreover, $\mu^{+,i} = S_{\bar{X}_i}(\mu^i)$ where μ^i is some minimizer of $E(T, \varphi_i)$. Let now $T \geq 1/4$. Since $\varphi_i \leq 1$, we have $\varphi_i^{-3/2}T \geq 1/4$ and thus by Corollary 1.2,

$$\mathcal{E}^+(\mu) = \frac{1}{2 - \sqrt{2}} \sum_{i=1}^N \varphi_i^{3/2} + NT.$$

For fixed N , this is minimized by $\varphi_i = 1/N$ so that

$$\mathcal{E}^+(\mu) = \frac{1}{2 - \sqrt{2}} N^{-1/2} + NT.$$

The function $x \rightarrow \frac{1}{2 - \sqrt{2}} x^{-1/2} + xT$ is minimized by $x_{opt} = (2T(2 - \sqrt{2}))^{-2/3}$. Since $x_{opt} < 2$ for $T \geq \frac{1}{4(2\sqrt{2}-2)}$ and $3 > x_{opt} > 2$ for $\frac{1}{4} \leq T < \frac{1}{4(2\sqrt{2}-2)}$, this concludes the proof. \square

As already explained in the introduction, this theorem is not completely satisfactory. Indeed, physically, the most significant case is $T \ll 1$ (where many microstructures should appear), which is not covered by Theorem 4.1. However, if we could prove the following conjecture,

Conjecture

For $T < T_*$ every minimizer of $E(T)$ branches at time zero (or equivalently $E(T - \varepsilon) < E(T) - \varepsilon$ for ε small enough),

then the picture would be almost complete. Indeed, in that case, arguing as in the proof of Theorem 4.1, we would have that every symmetric minimizer μ with $\mu_0 = \sum_{i=1}^N \varphi_i \delta_{X_i}$ would satisfy (4.2). Now for $1 \leq i \leq N$, let $\varphi_{i,1}, \dots, \varphi_{i,N_i}$ be the N_i branches starting from $(0, X_i)$. As in the proof of (3.4), we would have

$$E(T, \varphi_i) = \sum_{k=1}^{N_i} E(T, \varphi_{i,k}, \bar{X}_{i,k}),$$

where $\bar{X}_{i,k} := -\frac{1}{2} + \sum_{j < k} \varphi_{i,j} + \frac{\varphi_{i,k}}{2}$. Since the minimizer corresponding to $E(T, \varphi_{i,k}, \bar{X}_{i,k})$ cannot branch at time zero, we would have (if the conjecture holds) that $\varphi_{i,k}^{-3/2} T \geq 1/4$ so that Corollary 1.2 applies and the structure of the minimizers would be fully determined (recall also Lemma 2.8). Let us point out that our conjecture would be for instance implied by the convexity of $T \rightarrow E(T)$.

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